

On Locality of Generalized Reed-Muller Codes over the Broadcast Erasure Channel

Amira Alloum, Sian-Jheng Lin, and Tareq Y. Al-Naffouri

Abstract—One to Many communications are expected to be among the killer applications for the currently discussed 5G standard. The usage of coding mechanisms is impacting broadcasting standard quality, as coding is involved at several levels of the stack, and more specifically at the application layer where Rateless, LDPC, Reed Solomon codes and network coding schemes have been extensively studied, optimized and standardized in the past. Beyond reusing, extending or adapting existing application layer packet coding mechanisms based on previous schemes and designed for the foregoing LTE or other broadcasting standards; our purpose is to investigate the use of Generalized Reed Muller codes and the value of their locality property in their progressive decoding for Broadcast/Multicast communication schemes with real time video delivery. Our results are meant to bring insight into the use of locally decodable codes in Broadcasting.

Index Terms—Generalized Reed Muller codes, Erasure Channel, Application layer codes, Locally Decodable Codes.

I. INTRODUCTION: MOTIVATION AND RELATED WORK

In various Broadcasting standard reliable communication is achieved thanks to the use of coding mechanisms at different levels of the network layers either for overcoming the channel impairments or recovering lost symbols and packets. At the physical layer incremental redundancy HARQ mechanisms have been thoroughly studied using LDPC codes [1], and also standardized in UMTS and LTE using Turbo codes [2]. At upper layers historical Reed-Solomon (RS) codes have been first used in one to many communication standards as DVB-SH thanks to their MDS property. After the advent of LT and Raptor codes [3], rateless codes have been widely adopted as a reference for upper layer forward error correction (FEC) and specified for many Broadcast standards like DVB-SH, LTE-EMBMS. Their success is technically motivated by their universality as they are optimal over all erasure channels discarding any constraint of feedback. Another line of work proposed the use of Network Coding for packet loss concealment in Broadcast applications, where systematic binary network coding has been proved to be the best tradeoff in terms of complexity, throughput and decoding delay compared to classical straightforward network coding in finite fields [4]. The interest of combining multilayer coding has been analyzed in [5] and is revealed to be more beneficial for the multicast scenario than the point to point communication usecase.

Amira Alloum is with Nokia Bell Laboratories, France; Sian-Jheng Lin is with School of Information Science and Technology, University of Science and Technology of China (USTC), China and the Electrical Engineering Department, KAUST; Tareq Y. Al-Naffouri is with the Electrical Engineering Department, King Abdullah University of Science and Technology (KAUST).

Besides some broadcasting standards like MBMS specify a uni-cast repair mechanism after a limited broadcast delivery phase. During this phase, coding can be used on the top of TCP protocol for uni-cast communication where network codes or rateless codes can be used on top of physical layer in a separate or a cross layer approach [6].

In the context of recovering the packets or symbols lost during bad channel realizations, the common channel model considered to evaluate the recovery performance is the symbol (bit or block) erasure channel. On another side, most proofs derived for demonstrating capacity achieving properties have been derived over the erasure channel first before their extensions to other Binary Memoryless Channels (BMC).

While the common belief that random constructions are the unique road for capacity and that deterministic structures might be unable to honor the bet. This common belief has been abolished with the discovery of polar codes which are based on Reed Muller codes constructions. Consequently the invention of Polar codes rekindled the flame for algebraic codes among coding theory community, especially for Reed Muller Codes that have been introduced 50 years ago by Reed and Muller [7] [8], then have been generalized by Kasami and Lin in [9]. This class of code is special for having kept in the meanwhile the common interest of theoretical computer science and information theory communities concurrently. More recently Reed Muller codes have been proved to be capacity achieving over the binary and block erasure channel [10] [11].

Reed Muller codes are appreciated for being good practical extended cyclic codes meeting BCH codes in some instances of their generalization; they exhibit good geometrical and nesting properties and are good basis for constructing other codes, as they are a part of the LTE standard with the encoding of channel quality control informations [2] and have been used for Power Control issues for OFDM. Besides, Reed Muller codes exhibit an additional property called Locality which has been leveraged recently for coded based unconditionally secure protocols considered in theoretical computer science and cryptography communities [12]. Locality feature consist in the ability of retrieving a particular symbol of a coded message by looking only at $\ell < K$ positions of its encoding, where ℓ is known as the locality parameter and K denotes the dimension of the code. Besides, Locally decodable codes have been mentioned in [13] as a potential candidate for improving the power budget of HARQ schemes.

The recent results cited above, led us to rise some questions regarding the practical use of Reed Muller Codes in upper layer coding mechanisms and more specifically questioning about the value of locality in this context.

Our focus is on broadcasting for streaming services, where are required low complexity decoding algorithms in order to enable reception for energy constrained devices, together with short decoding delays in order to trigger the content reception as soon as possible without sacrificing throughput. In previous studies the decoding delay reduction is reached thanks to the use of systematic encoding constructions with progressive decoding. The complexity is decreased by the use of binary fields for network coding [4].

In our scheme we consider the decoding and delay performance of Generalized Reed Muller (GRM) codes over the block erasure channel under a locally symbol wise decoding algorithm. This is motivated by the fact that the GRM codes can be systematically encoded, and enable a progressive decoding thanks to the locality property earlier than word-wise Reed Solomon algebraic decoding.

The main purpose of our analysis, is to investigate the costs and the benefits of considering the locality in packets recovery using Generalized Reed Muller codes.

The paper is organized as follows: In section II we introduce the model and notations related to the Generalized Reed Muller Codes, Maximum likelihood (ML) decoder is described in section III, the exhaustive local decoder (LD) is derived and described in section IV, complexities of both ML and LD schemes are evaluated in section V. We show simulations results and discuss the decoding performances in section VI and conclude in section VII.

II. MODELS, NOTATIONS AND PROBLEM FORMULATION

A. Transmission and Channel Model

We consider a broadcasting communication standard where the sender is a basestation or a satellite transmitting k data packets encoded to k systematic packets and $(n - k)$ parity packets using an error correcting code.

The receivers are devices either within a satellite network or wireless cellular network, where each packet is considered by the upper layers either completely received or completely lost based on a CRC (Cyclic Redundancy Check) mechanism.

Accordingly the described practical scenario is modelled as a transmission occurring through a virtual block erasure channel characterised by an erasure probability ϵ over coded symbols in the considered finite field \mathbf{F}_q . Consequently each erased packet is related to an erased coded symbol of \mathbf{F}_q within the model.

B. Generalized Reed Muller Codes

Generalized Reed Muller code are constructed by complete evaluation of low degree multivariate polynomial over a finite field. The code is specified by parameters (r, m, q) where:

- q is the alphabet size and is a prime power.
- m denotes the number of variables.
- $r \leq q - 2$ is the degree of the polynomial.

Let $\mathbf{F}_q = \{\gamma_i\}_{i=0}^{q-1}$ denotes a finite field with q elements. Let us consider the variable vector $\mathbf{X} = (X_1, \dots, X_m)$ where $F(\mathbf{X}) \in \mathbf{F}_q[\mathbf{X}]$ denotes an m -variable polynomial of degree at most $r \leq q - 2$. The number of coefficients of $F(\mathbf{X})$ is the dimension of the code $k = \binom{m+r}{r}$ where the minimum

distance is $d = (q - r)q^{m-1}$ and $n = q^m$ is the code length of the code [14].

The (r, m) Reed-Muller code over \mathbf{F}_q is defined as:

$$(F(\mathbf{P}_1), F(\mathbf{P}_2), \dots, F(\mathbf{P}_n)),$$

where each point $\mathbf{P}_i \in \mathbf{F}_q^m$ belongs to the m -dimensional affine space over \mathbf{F}_q . Let us consider the vector:

$$V = [v_0 \dots v_{m-1}]^T \in \mathbf{F}_q^m \setminus \{0\}, \quad (1)$$

and the point:

$$P = [p_0 \dots p_{m-1}]^T \in \mathbf{F}_q^m. \quad (2)$$

The q elements:

$$y_i = F(P + \gamma_i \cdot V) = F_{P,V}(\gamma_i), \quad \gamma_i \in \mathbf{F}_q, \quad (3)$$

forms a $(q, r+1)$ Reed-Solomon (RS) codeword of dimension $r+1$ and code length q , and $F_{P,V}$ is a univariate polynomial of degree at most r .

Local decoding is possible over a block erasure channel, when $r+1$ received symbols are the evaluations of points aligned on a such a line, then the RS decoding algorithm can occur on the received symbol and perfectly reconstruct the erased $q - (r+1)$ symbols via polynomial interpolations.

III. MAXIMUM LIKELIHOOD DECODING FOR GENERALIZED REED MULLER CODES

The Maximum likelihood decoder over erasure channel fails to recover a given erasure pattern if this pattern contains the support of at least one non zero codewords. Their decoding consist in a Gaussian Elimination algorithm over the subsequent linear system.

The generator matrix G of GRM code is of size k -by- n , where the i -line can be determined with the RM encoder. Let $I_i = [0 \dots 0 \ 1 \ 0 \dots 0]$ denote a k -element zero vector with 1 at i -th position. By applying the RM encoding approach on I_i , the codeword vector, denoted as R_i , is the i -th row of G . If the code is systematic, the generator matrix is in the form $G = [I_k | \mathcal{P}]$, and its corresponding parity-check matrix is written as $H = [-\mathcal{P}^T | I_{n-k}]$.

Given a sent codeword C from the GRM codebook and a received codeword Y at the output of a block erasure channel, we consider the syndrome constraint $HY^T = 0$ from which we derive the following linear system :

$$H_0 Y_0^T = D^T, \quad (4)$$

where Y_0 denotes the erasure symbols, and H_0 consists of the subsequent columns corresponding to Y_0 . Our objective is to solve \tilde{Y}_0 in (4).

To solve (4), we apply Gaussian elimination on $[H_0 | D^T]$, to obtain a matrix \mathcal{Q} in the reduced row echelon form. For each row of \mathcal{Q} , if the row is in the form $[I_i | d_i]$, then the value of the i -th symbol is d_i . Gaussian elimination utilizes all parity equations in RM codes, so it performs the optimal performance for erasure channels. However, the algorithm is very slow and thus it is untractable for long codes in finite fields of high orders.

IV. LOCAL DECODING FOR GENERALIZED REED MULLER CODES

In the present section we derive an exhaustive local decoding algorithm for GRM codes.

Let us Consider the set Φ of received symbols in \mathbf{F}_q evaluating the set of points Φ' in \mathbf{F}_q^m . Our algorithm consists in exhaustively and sequentially search in the space, all the lines including $r+1$ received symbols evaluating $r+1$ aligned points of Φ' in order to apply Reed Solomon interpolation algorithm. For thus, it is required to enumerate all $r+1$ received symbols defining a support for a $(q, r+1)$ Reed Solomon sub-code among all the RS subcodes nested in a Reed Muller codeword.

In order to perform the exhaustive enumeration of the lines in the space, we build a partition of the space similar to the one used for the construction of Projective Reed Muller codes [14]. Our approach paves the way to the following proposition:

Proposition 1: The support of Generalized Reed Muller codeword of codelength $n = q^m$ includes the support of $q^{m-1} \times \frac{q^m-1}{q-1}$ number of $(q, r+1)$ Reed Solomon subcodes.

Proof:

In order to enumeratethe lines $\mathbf{P} + \gamma_i \cdot \mathbf{V}$, one can consider $q^m - 1$ instances of \mathbf{V} and q^m instances of \mathbf{P} where the affine space would include $(q^m - 1) \times q^m$ lines. Though, this is not a tight bound as some lines are counted multiple times.

Let us consider the partition of the affine space \mathbf{F}_q^m where $\mathbf{V}_0 \cup \dots \cup \mathbf{V}_{m-1} = \mathbf{F}_q^m \setminus \{0\}$ and defined as :

$$\mathbf{V}_i = \{V_i = [0 \dots 0 \ 1 \ v_{i+1} \dots v_{m-1}]^T | v_j \in \mathbf{F}_q, \\ i \leq j \leq m-1, v_i \neq 0\}$$

where \mathbf{V}_i denotes a set of m -element vectors where the first i elements are zeros, and $v_i \neq 0$. The number of instances per subspace is $|\mathbf{V}_j| = q^{m-j-1}$. Accordingly, the number of possible vectors V is $q^{m-1} + q^{m-2} + \dots + 1 = \frac{q^m-1}{q-1}$.

Let us consider the partition of $P \in \mathbf{F}_q^m$. The set of evaluation points is:

$$P + \gamma_i \cdot V_j = P - p_i V_j + (\gamma_i + p_i) \cdot V_j = P'_i + \gamma'_i \cdot V_j$$

where $P'_i = P - p_i V_j = [p'_0 \dots p'_{m-1}]^T$ is a vector with i -th element $p'_i = 0$. the set of values of P'_i is:

$$\mathbf{P}_i = \{[p_0 \dots p_{m-1}]^T | p_j \in \mathbf{F}_q, 0 \leq j \leq m-1, j \neq i, p_i = 0\}, \\ \text{and } |\mathbf{P}_i| = q^{m-1}.$$

Consequently we have enumerated $q^{m-1} \times \frac{q^m-1}{q-1}$ different lines in the affine space \mathbf{F}_q^m and so as much Reed Solomon subcodes nested in the space of an RM codeword. \square

Notably, when $r = q - 2$, the local decoding considers a $(q, q-1)$ RS codeword of one parity symbol. In this case, the set of received symbols of (3) satisfy the parity equation

$$F_{P,V}(\gamma_0) + \dots + F_{P,V}(\gamma_{q-1}) = 0.$$

Thus, we can decode one symbol via only field additions, and the field multiplications are unnecessary. Based on the previous discussion, the sequential exhaustive local decoding algorithm is described in Algorithm 1.

Algorithm 1: Decoding Reed-Muller codes via locally decoding approach

Data: A set of RM symbols $\{(e_i, \mathbf{U}_i)\}_{i=0}^{\ell-1}$, where $e_i = F(\mathbf{U}_i)$ is the symbol and $\mathbf{U}_i \in \mathbf{F}_q^m$ indicates its evaluation point

Result: A codeword M' via locally decoding approach

```

1  z = 1
2  while z = 1 do
3      z = 0
4      for i = 0 to (m - 1) do
5          for V ∈ Fqi-1 do
6              V = [0 ... 0 1 V]
7              for P0 ∈ Fqm-i-2 do
8                  for P1 ∈ Fqi-1 do
9                      P = [P0 0 P1]
10                     Count the number of received symbols
11                     R on (P + γi · V), for γi ∈ Fq.
12                     if R ≥ r + 1 then
13                         Apply (q, r + 1) RS decoding to
14                         interpolate the lost symbols.
15                         z = 1
16                     end
17                 end
18             end
19         end
20     end
21 return M'

```

V. COMPLEXITY ANALYSIS

This section discusses the computational complexities of the local decoding algorithm (LD) and the Gaussian elimination (GE) algorithm described in the previous sections.

The maximum likelihood decoder is known to be the one of the GE applied on the matrix $[H_0 | D^T]$, which requires $O(n^3)$ operations. However regarding the proposed LD decoder the theoretical complexity is not straightforward, as the number of loops (line 1-19) is not analytically quantified so far.

Nevertheless, we introduce an alternative related decoding strategy, termed progressive local decoding (PLD), that triggers the decoding on the fly with the reception of a fraction of the codeword on a symbol by symbol basis .

Once a symbol is received, PLD checks all lines across this symbol. Once a line includes $r+1$ either received or recovered symbols, RS decoding is applied. For a codeword of (r, m, q) RM code, there are q^{m-1} lines across a symbol. With the use of fast fourier transform (FFTs) techniques, RS erasure decoding requires $O(q \lg(q))$ operations [15]. Hence, the global per symbol recovery operation requires $O(q^m \lg(q))$ operations in each round. As the decoder will receive at most n symbols, the complexity is quasi quadratic and no more than $O(nq^m \lg(q)) = O(n^2 \lg(q))$. Consequently the PLD achieves lower complexity than GE does by around one order.

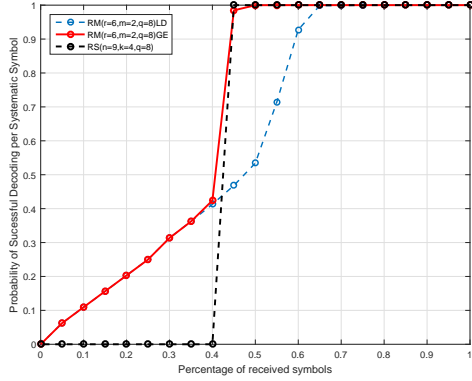


Fig. 1. Probability of Successful Decoding Vs percentage of received symbols.

VI. SIMULATION RESULTS AND DISCUSSION

In the present section we show simulation results assessing the local decoding (LD) performance of systematic GRM codes over the block erasure channel.

We have selected metrics emphasizing the potential of the local decoder for progressive packet recovery in streaming applications [4]. For instance the metric represented in Figure 1 is the probability of successful decoding per information symbol varying with percentage of received symbols. In this scheme we consider LD and GE decoding of the $(r = 6, m = 2, q = 8)$ RM code compared with interpolation decoding of the $(n = 9, k = 4, q = 8)$ RS code in \mathbb{F}_8 . We chose same fields orders to align erasure dimension over the channel.

The results in Figure 1 show that GE and local decoding execute partial decoding before the reception of k symbols and before building a full rank linear system for the GE. The locality being $r + 1 = 7$ symbols, the local decoder triggers recovery when receiving 10 percent of transmitted symbols which is lower than the code dimension that is 43, 37 percent of the transmission as $k = 28$.

It is also exhibited that LD under-performs the RS decoder when reception is beyond the code dimension. However GE performs as good as the RS decoder and better than LD decoding after building a full rank linear system, an erasure point that we refer to as the full rank threshold ϵ^* .

Figure 2 shows the computational cost in terms of time per codeword varying with the percentage of erased symbols. In line with the theoretical analysis we find out that the GE decoding requires more operations than the LD and that the complexity gap is getting increased with the field order.

Accordingly we conclude that LD is an alternative to the GE decoder for systematic locally decodable codes before the full rank threshold ϵ^* . The LD enhances the information recovery delay at lower complexity than ML decoder and with equal performance. However beyond ϵ^* the system should switch to the GE to get the best performance in a combined architecture LD-GE. A further comparison to systematic random network coding is worth to be investigated. Whereas GRM codes have the inherent benefit of being systematic, and locally decodable with a mitigated complexity.

VII. CONCLUSIONS AND FURTHER WORK

In this paper, we evaluated the value of locality property in GRM codes for packet recovery in Broadcast Applications. We revealed that locality and systematic encoding of GRM codes are valuable assets for progressive decoding in streaming

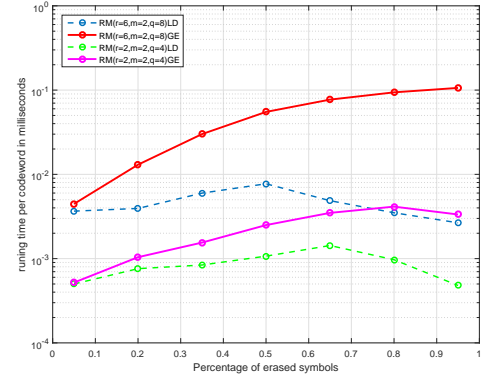


Fig. 2. Running time for RM($r=6, m=2, q=8$), RM($r=2, m=2, q=4$).

usecases. However, among the drawbacks we should mention about GRM codes is their decreasing rate when code-length get longer. Therefore investigating other locally decodable constructions at higher rates may be an interesting future research avenue for broadcasting applications.

REFERENCES

- [1] E. Soljanin, N. Varnica, and P. Whiting, "Incremental redundancy hybrid ARQ with LDPC and raptor codes," *IEEE Transactions on Information Theory*, 2005.
- [2] 3GPP TS 36.212 *Evolved Universal Terrestrial Radio Access (E-UTRA), Multiplexing and Channel Coding, Rel-13*, Std., 2015.
- [3] A. Shokrollahi, "Raptor codes," *IEEE Transactions on Information Theory*, vol. 52, no. 6, pp. 2551–2567, 2006.
- [4] A. L. Jones, I. Chatzigeorgiou, and A. Tassi, "Binary systematic network coding for progressive packet decoding," in *Communications (ICC), International Conference on*. IEEE, 2015, pp. 4499–4504.
- [5] M. Heindlmaier and E. Soljanin, "Isn't hybrid arq sufficient?" *arXiv preprint arXiv:1411.4061*, 2014.
- [6] J. K. Sundararajan, D. Shah, M. Médard, M. Mitzenmacher, and J. A. Barros, "Network coding meets TCP," in *INFOCOM*. IEEE, 2009, pp. 280–288.
- [7] I. Reed, "A class of multiple-error-correcting codes and the decoding scheme," *Information Theory, Transactions of the IRE Professional Group on*, vol. 4, no. 4, pp. 38–49, September 1954.
- [8] D. E. Muller, *Metric properties of Boolean algebra and their application to switching circuits*, 1953.
- [9] T. Kasami, S. Lin, and W. W. Peterson, "New generalizations of the Reed-Muller codes-i: Primitive codes," *IEEE Transactions on Information Theory*, vol. 14, no. 2, pp. 189–199, 1968.
- [10] S. Kumar and H. D. Pfister, "Reed-Muller codes achieve capacity on erasure channels," *arXiv preprint arXiv:1505.05123*, 2015.
- [11] S. Kudekar, M. Mondelli, E. Şaşoğlu, and R. Urbanke, "Reed-Muller codes achieve capacity on the binary erasure channel under MAP decoding," *arXiv preprint arXiv:1505.05831*, 2015.
- [12] S. Yekhanin, "Locally decodable codes," in *Computer Science—Theory and Applications*. Springer, 2011, pp. 289–290.
- [13] C. Lott, O. Milenkovic, and E. Soljanin, "Hybrid arq: theory, state of the art and future directions," in *Information Theory Workshop on Information Theory for Wireless Networks*. IEEE, 2007, pp. 1–5.
- [14] G. Lachaud, "Projective reed-muller codes," in *Coding Theory and Applications*. Springer, 1986, pp. 125–129.
- [15] S.-J. Lin, W.-H. Chung, and Y. S. Han, "Novel polynomial basis and its application to reed-solomon erasure codes," in *The 55th Annual Symposium on Foundations of Computer Science (FOCS'14)*, 2014.